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Optimal Control of an Assembly System with Demand for the End-Product and Intermediate Components

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September 2009, revised September 2010, February 2011

Abstract

We consider the production and admission control decisions for a two-stage manufacturing system where intermediate components are produced to stock in the first stage and an end-product is assembled from these components through a second stage assembly operation. The firm faces two types of demand. The demand for the end-product is satisfied immediately if there are available products in inventory while the firm has the option to accept the order for later delivery or to reject it when no inventory is available. Demand for intermediate components may be accepted or rejected to keep components available for assembly purposes. We characterize the structure of demand admission, component production and product assembly decisions. We also extend the model to take into account multiple customer classes and a more general revenue collecting scheme where only an upfront partial payment is collected if a customer demand is accepted for future delivery with the remaining revenue received upon delivery. Since the optimal policy structure is rather complex and defined by switching surfaces in a multidimensional space, we also propose a simple heuristic policy for which the computational load grows linearly with the number of products and test its performance under a variety of example problems.

Keywords: Production and inventory control, Multistage assembly systems, Inventory rationing, Demand management, Markov decision processes, Dynamic programming, Optimal control

1 Introduction

This paper focuses on a manufacturing setting where a firm, having both component production and final assembly operations, faces demands for its end-product as well as the intermediate components.

Several business practices may lead a firm to operate within this setting. For example, consider a major appliance manufacturer such as Whirlpool, that produces various components and assembles them into a refrigerator. In addition to the demand for the refrigerator, Whirlpool also supplies individual components such as compressors in order to sustain its after-sales service operations. In many instances, efficient production control and demand management skills may be crucial for profitability when the product and component sales both have a significant contribution to the firm's revenues. The after-sales service is regarded as a high profit margin business and has become a comparable revenue generator throughout numerous industries. According to Cohen et al (2006), in industries such as automobiles and white goods, the earlier units that companies have sold over the years have created aftermarkets four to five times larger than the original product markets. Consequently, businesses across many industries earn on average 45% of gross profits from the aftermarket.

As another example, consider TRW, which produces a range of automotive components such as braking, steering and suspension systems. TRW has a unit that makes engineered fastener components for its own products but the fastener unit also sells fasteners to other Tier 1 automotive suppliers which may sometimes even compete with TRW. Thus, at any point in time, the fastener unit has the option to accept or reject outside demand but also needs to coordinate its production policies to serve the demand arising from the assembly of its own products.

Our main objective in this paper is to address several important decisions that a firm operating within this setting faces. Specifically, we will be focusing on the following questions: 1) How many of each type of intermediate components should the firm produce? 2) How should the firm decide whether to accept or reject an order for any of these intermediate components? 3) How does the firm determine whether to initiate the assembly of another end-product? 4) How does the firm regulate end-product admissions to its assembly queue?

This study interconnects the two research areas of assembly and admission control. There exists a rich literature on inventory control of assembly systems. An extensive literature survey has been provided by Song and Zipkin (2003). In one of the earliest works, Schmidt and Nahmias (1985) study an assembly system with two components and a single final product that is assembled-to-stock. They assume a two stage manufacturing system where both the production and assembly stages have deterministic lead times. They identify the optimal assembly policy which states that there exists a target assemble-up-to point to reach as long as there are available components. They also identify the optimal production policies for the components which follows a modified base-stock policy due to differing replenishment lead times for the components. Rosling (1989) extends the findings of Schmidt and Nahmias (1985) to multi-stage assembly systems by also assuming deterministic lead times.

In more recent works on pure assembly systems which are closest to our setting, Benjaafar and ElHafsi (2006) consider the production and inventory control of a multi-component assembly system with several customer classes. Assuming instantaneous assembly, they show that a state dependent base stock policy is optimal for component production and there exists state dependent rationing levels for different demand classes. A subsequent work by Benjaafar et al (2010) incorporates multiple production and inventory stages in an assembly operation. They again characterize the structure of the optimal production and rationing policies in the presence of multiple customer classes and show that production at each stage follows a state dependent base-stock policy which decreases with the inventory level of downstream items and increases with the inventory level of all other items. As in their previous work, demand admission for the product follows state dependent rationing levels.

In Benjaafar et al. (2010), all customer classes require the same end-product but are willing to pay different amounts for the product. Therefore, rationing decisions are taken at the end product inventory level to prioritize several demand streams for the same product. In our setting, different customers demand different products, either the end-product or any of the intermediate components.

The majority of the work on assembly systems considers the time to assemble a product is often negligible compared to the production time of components, and consequently, they assume instantaneous assembly. In our model, we relax the assumption that assembly is instantaneous, enriching our setting in important aspects. For example, this relaxation enables us to identify the economic benefits of having the components turned into a final product versus keeping them available for individual sales by considering the important decision of when to initiate an assembly.

There is also a rich literature on admission control which falls outside the assembly systems classification. Stidham (1985) presents a review of the literature on admission control for a single class make-to-order queue. Ha (1997a) considers a single item make-to-stock production system with several demand classes and lost sales. He shows that the optimal admission control policy is characterized by stock rationing levels for each demand class. He later extends the results to allow backorders in Ha (1997b) for a make-to-stock production system with two priority classes.

Specifically, controlled arrival to multiple nodes of queues in series has also attracted interest. Ghoneim and Stidham (1985) study one such setting with two queues in series where customer arrivals to the first queue go through service in both queues whereas customer arrivals to the second queue only require service by that queue. They show that the optimal demand admission policy has a monotonic structure. Ku and Jordan (1997) also study a similar system with finite queue sizes. They introduce randomness on whether a customer admitted to the first queue will actually stay in the system to get service from the second queue. They show that the optimal admission policy is defined by a monotonic threshold. In a subsequent work, Ku and Jordan (2002), extend their results to systems with parallel first-stage queues. Duenyas and Tsai (2001) study a two-stage production/inventory system where there is demand for the end product as well as the intermediate product with admission control on the demand for the latter. Their setting incorporating admission control for the intermediate product is an extension to the problem studied by Veatch and Wein (1994), which focuses on the control of a two-stage tandem production system. Duenyas and Tsai (2001) derive the structure of the optimal policy for the centralized control problem and consider several pricing schemes for the decentralized case that achieves the profits of the centralized problem. Their formulation for the centralized control problem where there is only a single component and no admission control on the end-product is a special case of the problem considered in this paper which considers a general multi product setting through a two-stage production and assembly system.

Our work is also related to the general assemble-to-order manufacturing systems literature involving multiple products assembled from a selection of intermediate components. As stated in Song and Zipkin (2003), optimal policies regarding such general systems are still unknown. Several authors have focused on control policies assuming an independent base-stock order policy along with some allocation rule, such as committing inventories to the earliest backlog or simply following a first come first serve allocation. Examples of such work include those of Hausman et al (1998), Song et al (1999), and Akcay and Xu (2004). Our formulation may be regarded as a special case of the general system in the sense that it only allows demands - in addition to the demand for the end product - for one component at a time rather than an arbitrary selection of multiple components. Albeit limited compared to a general product portfolio architecture, our model enables us to characterize the structure of the optimal policies.

The remainder of the paper is organized as follows. In Section 2, we provide the problem formulation. In Section 3, we characterize the structure of the optimal policies for the production and rationing decisions regarding individual components and the assembly and admission decisions for the end-product. We investigate various sensitivity characteristics of the optimal policy and discuss extensions to the original model. In Section 4, we provide numerical results to evaluate the performance of a heuristic solution tested for a variety of problem instances. We conclude in Section 5.

2 Problem Formulation

We consider a two-stage assembly system as shown in Figure 1. In the first stage, N intermediate components (also referred to as intermediate products) are produced-to-stock in exclusive subassembly lines and in the second stage, they are assembled into a single end-product. There are two types of demand sources in the system. The first type of demand is for the end-product, arriving based on a Poisson distribution with rate λ_0. The second type of demand is directly for the intermediate components. Customers requesting a specific component i also arrive following a Poisson process with a demand rate λ_i, $i = 1, 2, ...N$. As a side note, for after-sales operations

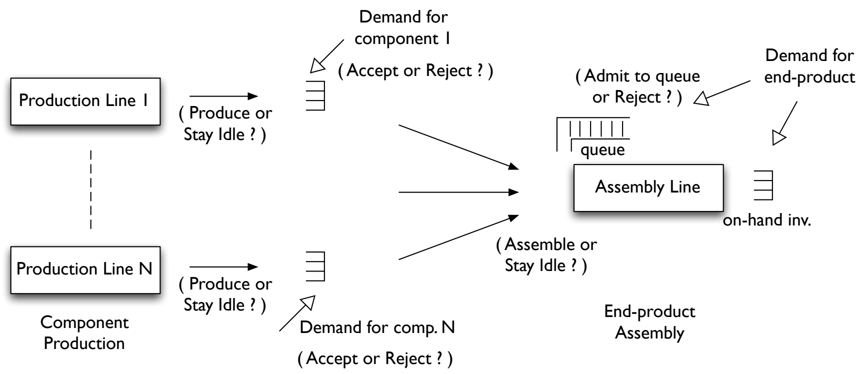

Figure 1: The assembly system demonstrating the demand for intermediate components as well as the end-product and the corresponding decisions.

that were mentioned in the motivation section, the demand arrival rate for each component in the future may be positively correlated with the current sales of end-products (which itself is influenced by both the end-product demand rate as well as the control policies used by the firm). However, to preserve the tractability of our analysis, we ignore such correlation effects.

We assume that production of a unit of component i ($i = 1, 2, ...N$) takes an exponentially distributed amount of time with mean $\frac{1}{\mu_i}$. The subassembly lines feed a single downstream assembly line, referred to as the second stage of the manufacturing system. During this assembly stage, one unit of each type of component is drawn from its inventory, and assembled into a single end-product. We let the assembly operation for the product also take an exponentially distributed amount of time with mean $\frac{1}{\mu_0}$.

Let the state $(\mathbf{X}(t), Y(t)) \in S$ where $\mathbf{X}(t) = (X_1(t), X_2(t), ..., X_N(t))$ and $X_i(t) \geq 0, \forall\ i = 1, ..., N$ be defined such that $X_1(t), X_2(t), ..., X_N(t)$ denote the amount of inventory of components 1 through N respectively, and $Y(t)$ denotes the inventory position for the end-product at time t. The states at which $Y(t) < 0$ correspond to customer orders waiting in the assembly queue whereas states at which $Y(t) > 0$ indicate that there is available inventory ready to satisfy end-product demand. The firm incurs inventory holding and backorder costs at the rate $c(\mathbf{X}(t), Y(t)) = \sum_{i=1}^{N} h_i X_i(t) + h_0 Y^+(t) + b_0 Y^-(t)$ where $Y^+(t) := \max(Y(t), 0)$, $Y^-(t) := -\min(Y(t), 0)$, h_i is the unit holding cost for each component i kept in stock, h_0 and b_0 is the unit holding and backorder cost for each product kept in stock and order kept in the assembly queue, respectively. We note that preemptions are allowed during both the assembly and production operations. In addition, we exclude the cost of subassembly and assembly operations from the model since, without loss of generality, we can define the revenues from individual component and end-product demands as marginal revenues.

The decision epochs considered in this model consists of all demand arrivals together with the production and assembly completions. At each decision epoch, a policy specifies whether a production server should stay idle or produce a unit of the corresponding component and whether the assembly line should stay idle or initiate the assembly of another end-product. At decision epochs corresponding to demand arrivals for the end-product or to individual components, the policy determines whether to accept or reject the orders.

The profit is the revenue from accepted orders minus the inventory holding and backorder costs. The goal is to find a policy which maximizes the following average profit per unit time:

$$\lim_{T\to\infty} \inf \frac{1}{T} E^{\pi} \left[\int_0^T -c(\mathbf{X}(t), Y(t))dt + \sum_{i=0}^{N} R_i N_i(T) \right] \tag{1}$$

where the expectation is taken under a policy π and $N_i(T)$ denotes the number of orders of each component ($i = 1, ..., N$) and the product ($i = 0$) that has been accepted up to time T.

Following Lippman (1975), we analyze a uniformized version of the problem given in (1) by defining a transition rate $\Lambda = \sum_{i=0}^{N} (\lambda_i + \mu_i)$. We let $v(\mathbf{x}, y)$ denote the relative value function of being in state $(\mathbf{x}, y)$ and g be the average profit per transition (see proof of Theorem 1 for the existence of an average profit), resulting in an average profit per unit time of $g\Lambda$.

Before proceeding further, we first introduce a set of operators to represent the firm's decisions in order to simplify the presentation of the formulation and to assist us in the analysis to follow.

First, we will consider the end-product demand admission decision. When an order for the end-product arrives, if the end-product inventory is positive, the order is met immediately from inventory. However, if there is no available end-product inventory, the firm has the option to accept or reject the order. Each accepted order generates a revenue of R_0. If an order is rejected, it is considered as lost sales. If the end-product inventory is zero, and we accept the order, we incur a backorder cost. Operator T_0^1 represents this decision regarding product admission.

$$T_0^1 v(\mathbf{x}, y) = \begin{cases} v(\mathbf{x}, y-1) + R_0 & \text{if } y > 0 \\ \max[v(\mathbf{x}, y-1) + R_0, v(\mathbf{x}, y)] & \text{if } y \leq 0 \end{cases}$$

Similarly, there is an admission decision associated with each demand arrival for any of the intermediate components. Acceptance of a demand for a specific component i leads to a revenue of R_i whereas rejection of an order leads to lost sales. Since components are produced to stock, an order for a component may be accepted only if the inventory of the corresponding component is positive. Thus, component demand cannot be backordered. The operator T_i^1 defined below corresponds to the component demand admission decision where $I_{(\cdot)}$ denotes the indicator function and $\mathbf{e}_i$ is the i^{th} unit vector.

$$T_i^1 v(\mathbf{x}, y) = \max\left[(v(\mathbf{x} - \mathbf{e}_i, y) + R_i) \cdot I_{(x_i>0)} + v(\mathbf{x}, y) \cdot I_{(x_i=0)}, v(\mathbf{x}, y)\right]$$

Finally, operators T_0^2 and T_i^2 defined below correspond to the assembly initiation and component production decisions, respectively.

$$T_0^2 v(\mathbf{x}, y) = \max\left[(v(\mathbf{x} - \mathbf{1}, y + 1)) \cdot I_{(x_i>0\ \forall i)} + v(\mathbf{x}, y) \cdot I_{(\exists\ i\ |\ x_i=0)}, v(\mathbf{x}, y)\right]$$

$$T_i^2 v(\mathbf{x}, y) = \max[v(\mathbf{x} + \mathbf{e}_i, y), v(\mathbf{x}, y)]$$

We now present the average profit infinite horizon dynamic programming formulation.

$$v(\mathbf{x}, y) + g = \frac{1}{\Lambda}\left(-c(\mathbf{x}, y) + \sum_{i=0}^{N}\left(\lambda_i\ T_i^1\ v(\mathbf{x}, y) + \mu_i\ T_i^2\ v(\mathbf{x}, y)\right)\right) \tag{2}$$

In subsequent sections, we refer to the right hand side of (2) by operator T defined such that $v(\mathbf{x}, y) + g = Tv(\mathbf{x}, y)$. In (2), the terms $\frac{1}{\Lambda}(c(\mathbf{x}, y))$ denote, respectively, the expected costs per decision epoch due to holding of component inventories, holding of end-product inventories, and backordering customer orders in the queue. The terms multiplied by $\lambda_i, (i = 0, 1, 2, ..., N)$ correspond to transitions and revenues generated with the arrival of a demand for the end-product when i=0 and the components when i=1,...,N. Similarly, the terms multiplied by μ_i correspond to transitions and revenues generated by a product assembly when i=0 and by a component production completion opportunity when $i = 1, ...N$.

3 Characterization of the Optimal Policy Structure

3.1 Structure and Sensitivity of Optimal Production, Assembly and Admission Policies

In this section, we characterize the optimal production, assembly and admission policies. The main questions of interest are the following: (1) Should the firm produce an additional unit of a component or not? (2) If a demand arrives for an individual component, should this demand be satisfied or rejected in order to keep the components available for the end-product assembly? (3) When there are available components, should another unit of an end-product be assembled? (4) When a demand arrives for the end-product while no inventory is available, should the firm admit the demand to the assembly queue or reject it?

First, we introduce the following difference operators that will facilitate the characterization of the optimal policy structure. For any real valued function v on the state space, we define:

$D_i v(\mathbf{x}, y) = v(\mathbf{x} + \mathbf{e}_i, y) - v(\mathbf{x}, y) \;\; \forall\, i = 1, \ldots, N,$
$D_p v(\mathbf{x}, y) = v(\mathbf{x}, y + 1) - v(\mathbf{x}, y),$
$D_{\text{-}\mathbf{1},p} v(\mathbf{x}, y) = v(\mathbf{x}, y + 1) - v(\mathbf{x} + \mathbf{1}, y).$

D_i represents the additional value of an additional unit of component type-i inventory under value function v. D_p is the additional value of an additional unit of end-product inventory. Finally, $D_{\text{-}\mathbf{1},p}$ refers to the value of having an additional unit of an end-product relative to the value of keeping the components in component inventories.

Let V be the set of functions defined on the state space such that if $v \in V$, then $\forall\, i, j = 1, \ldots N$ where $j \neq i$;

(i) $D_i v(\mathbf{x}, y) \;\downarrow x_i, \uparrow x_j, \downarrow y \;\; \forall\, i = 1, \ldots, N$
(ii) $D_p v(\mathbf{x}, y) \;\downarrow x_i, \downarrow y$, and $\leq R_0$ for $y > 0$
(iii) $D_{\text{-}\mathbf{1},p} v(\mathbf{x}, y) \;\uparrow x_i, \downarrow y$

The above conditions are the second-order monotonicity conditions on v and characterize the structure of the optimal component production and rationing policies. For example, in (i), $D_i \downarrow x_i$ indicates that $v(\mathbf{x}, y)$ is concave in each of the state variables x_i. Equivalently, this means that the additional value gained by producing a unit of component type-i gets smaller with each additional unit of component type-i inventory. Hence, if it is optimal not to produce component type-i in state $(\mathbf{x}, y)$, it remains optimal not to produce it in state $(\mathbf{x} + \mathbf{e}_i, y)$. This implies that if condition (i) holds, the component production policies follow state-dependent base-stock policies. The following conditions, $D_i \uparrow x_j$ and $D_i \downarrow y$ mean, respectively, that $v(\mathbf{x}, y)$ is supermodular in (x_i, x_j) and submodular in (x_i, y). These conditions imply that the base-stock level for component type-i is nondecreasing with the inventory of other components and nonincreasing with the end-product inventory. Consequently, since backorders for the end-product imply a negative inventory position for this product, as the number of customers waiting in the assembly queue increases, the base-stock level for component type-i increases. In (ii), $D_p \downarrow y$ means that $v(\mathbf{x}, y)$ is concave in y. This implies that if condition (ii) holds, end-product admission policy will be of a threshold type. $D_p \downarrow x_i$ (which is equivalent to $D_i \downarrow y$) is a submodularity condition that implies that the admission threshold is nondecreasing with component inventories. Similarly, in (iii), the supermodularity condition $D_{\text{-}\mathbf{1},p} v(\mathbf{x}, y) \uparrow x_i$ implies that the additional value of assembling another end-product gets larger with each additional unit of component inventory, while the submodularity condition $D_{\text{-}\mathbf{1},p} v(\mathbf{x}, y) \downarrow y$ means that the additional value of assembling another end-product gets smaller with each additional unit of end-product in inventory.

We further introduce secondary difference operators followed by a set of additional sub- and super-modularity conditions that facilitate our derivation of the optimal policy structure.

$D_{\mathbf{1}}v(\mathbf{x},y) = v(\mathbf{x}+\mathbf{1},y) - v(\mathbf{x},y)$

$D_{i,j}v(\mathbf{x},y) = v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j,y) - v(\mathbf{x},y)\ \forall\ i,j = 1,\ldots,N$ where $j \neq i$

$D_{-i,p}v(\mathbf{x},y) = v(\mathbf{x},y+1) - v(\mathbf{x}+\mathbf{e}_i,y)$

$D_{i,-\mathbf{1},p}v(\mathbf{x},y) = v(\mathbf{x}+\mathbf{e}_i,y+1) - v(\mathbf{x}+\mathbf{1},y)$

Lemma 1. *If a value function v satisfies conditions* (i)-(iii)*, then v also satisfies the following conditions $\forall\ i,j,k = 1,\ldots N$ where i,j,k are distinct:*

(iv) $D_{\mathbf{1}}v(\mathbf{x},y) \downarrow x_i, \downarrow y$

(v) $D_{i,j}v(\mathbf{x},y) \downarrow x_i, \downarrow x_j, \uparrow x_k, \downarrow y$

(vi) $D_{-i,p}v(\mathbf{x},y) \uparrow x_i, \downarrow x_j, \downarrow y$

(vii) $D_{i,-\mathbf{1},p}v(\mathbf{x},y) \downarrow x_i, \uparrow x_j, \downarrow y$

Proof: The proof of Lemma 1 and all subsequent results are provided in the Appendix.

As Lemma 1 reveals, the above relations are implied solely by conditions (i)-(iii). Although they do not have direct ramifications on the optimal policy structure, their frequent appearances in the analysis of the following lemma warrant their universal treatment. Lemma 2 shows that the conditions (i)-(iii) are preserved under the operator T.

Lemma 2. *If $v \in V$ then, $T_0^1 v$, $T_i^1 v$, $T_0^2 v$, $T_i^2 v$, and $Tv \in V \quad \forall i = 1,\ldots,N$.*

We now present the main result. The policies defined below reflect the structure of the optimal policy for our model.

Definition 1. Consider the N-dimensional integer valued vectors $(\mathbf{x}_{-i},y)$ and $(\mathbf{x})$ where $\mathbf{x}_{-i}$ denotes the inventory level of all components except component type-i. Define the following (state-dependent) component rationing and product admission policies:

(a) Rationing policy for component type-i: A demand for component type-i is satisfied if the amount of inventory for the component is greater than or equal to a rationing threshold $\alpha_i(\mathbf{x}_{-i},y)$, i.e. if $x_i \geq \alpha_i(\mathbf{x}_{-i},y)$. If $x_i < \alpha_i(\mathbf{x}_{-i},y)$, the demand for the component is rejected.

(b) Admission policy for end-product: A demand for the end-product is admitted if the end-product inventory position is greater than or equal to an admission threshold $\beta(\mathbf{x})$, i.e. if $y \geq \beta(\mathbf{x})$. If $y < \beta(\mathbf{x})$, the end-product demand is rejected.

Definition 2. For the integer valued vectors $(\mathbf{x}_{-i},y)$ and $(\mathbf{x})$, define the following (state-dependent) base-stock policies for component production and product assembly:

(a) Base-stock policy for component type-i: An additional unit of component type-i is produced if the inventory for component type-i is less than a production threshold $\gamma_i(\mathbf{x}_{-i},y)$, i.e. if $x_i < \gamma_i(\mathbf{x}_{-i},y)$. If $x_i \geq \gamma_i(\mathbf{x}_{-i},y)$, the production resource for the component stays idle.

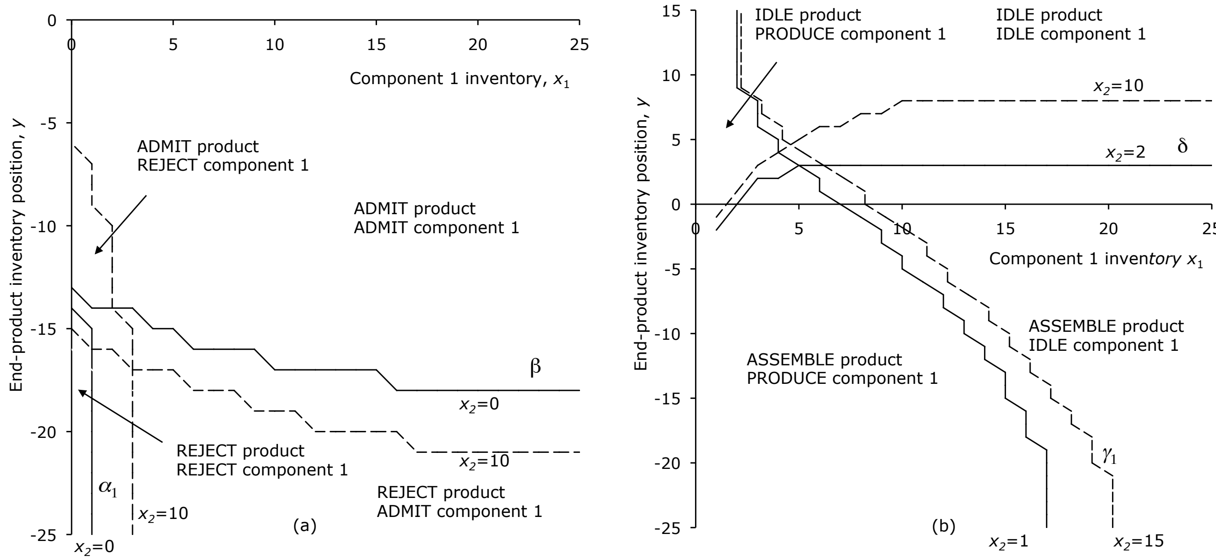

Figure 2: Structure of the optimal policies. (a) admission decisions for component type-1 and the end-product. (b) production decisions for component type-1 and the end-product.

(b) Base-stock policy for end-product: When all components are available, the assembly operation is initiated if the end-product inventory position is lower than an assembly threshold level $\delta(\mathbf{x})$, i.e. if $y < \delta(\mathbf{x})$. For $y \geq \delta(\mathbf{x})$, the assembly resource for the product stays idle.

The following theorem gives the characterization of the optimal policy structure.

Theorem 1. (a) *Demand admissions for each individual component type-i, $i = 1, \ldots, N$, follows a rationing policy characterized by the rationing threshold $\alpha_i(\boldsymbol{x}_{-i}, y)$. Furthermore, $\alpha_i(\boldsymbol{x}_{-i}, y)$ is non-decreasing with x_j, $j = 1,\ldots,N$, $j \neq i$, and non-increasing with y.*

(b) *Admissions for end-product demand follow an admission policy characterized by the threshold $\beta(\boldsymbol{x})$ which is non-increasing with x_i, $\forall i = 1,\ldots, N$.*

(c) *Production policy for each component type-i is defined by a base-stock policy with a production threshold $\gamma_i(\boldsymbol{x}_{-i}, y)$. Furthermore, $\gamma_i(\boldsymbol{x}_{-i}, y)$ is non-decreasing with x_j, $j = 1,\ldots,N$, $j \neq i$, and non-increasing with y.*

(d) *Assembly policy for the end-product follows a base-stock policy with an assembly threshold $\delta(\boldsymbol{x})$ which is non-decreasing with x_i, $\forall i = 1,\ldots, N$.*

Figure 2 illustrates the structure of the optimal policies described in Theorem 1 for an example problem with two components with the following parameters: $\lambda_0 = 5$, $\lambda_1 = 3$, $\lambda_2 = 4$, $\mu_0 = 8$, $\mu_1 = \mu_2 = 10$, $R_0 = 40$, $R_1 = 20$, $R_2 = 10$, $b_0 = 4$, $h_0 = 2$, and $h_1 = h_2 = 1$. The results are obtained by solving the average profit dynamic programming equation given in (2) using the value iteration algorithm (see, for example, Tijms (1986)). The value iteration algorithm is terminated

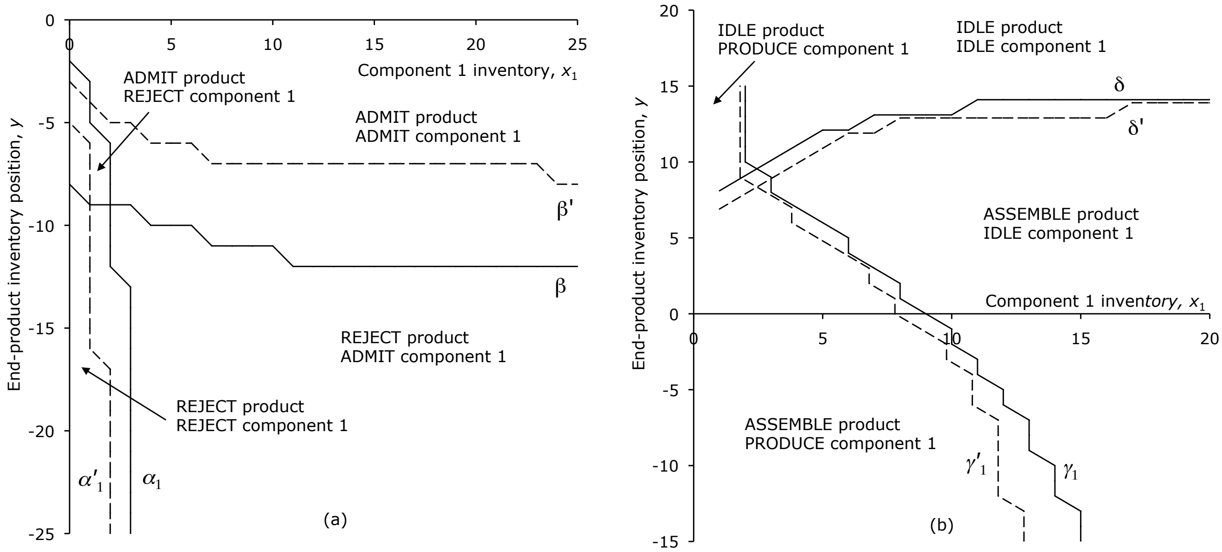

Figure 3: Changes in optimal policies due to a decrease in the end-product revenue: $R_0' < R_0$

when five-digit precision is attained. The resulting relative values for each state reveal the optimal control action corresponding to the demand admission, production and assembly decisions at that stage.

The switching curves α_1 and β in Figure 2 (a) depict the component rationing (for component type-1) and end-product admission threshold levels, respectively. Figure 2 (b) displays the structure of the optimal component production and end-product assembly policies represented by the switching curves γ_1 and δ, respectively. Similar switching curves exist for the type-2 component.

For this example problem consisting of two components, the optimal threshold values are defined by switching surfaces in three dimensions. The solid and the dotted curves in both figures are results of two-dimensional cuts on the switching surfaces at two separate values of the type-2 component inventory levels. For a general problem with N components, each threshold is defined by a switching surface embedded in an $N+1$ dimensional space.

Next, we will examine how the optimal policies described in Theorem 1 change as the end-product revenue decreases. We will use the prime symbol (') while referring to the relative value function and parameters of the modified problem.

Theorem 2. *Suppose that* $\mu_i' = \mu_i$, $\lambda_i' = \lambda_i$ *and* $h_i' = h_i$ *for i=0,1,...,N;* $b_0' = b_0$, *and* $R_i' = R_i$ *for* $i = 1, ..., N$ *whereas* $R_0' < R_0$. *Then,* $\alpha_i'(\boldsymbol{x}_{-i}, y) \leq \alpha_i(\boldsymbol{x}_{-i}, y)$, $\beta'(\boldsymbol{x}) \geq \beta(\boldsymbol{x})$, $\gamma_i'(\boldsymbol{x}_{-i}, y) \leq \gamma_i(\boldsymbol{x}_{-i}, y)$, *and* $\delta'(\boldsymbol{x}) \leq \delta(\boldsymbol{x})$.

Regarding the admission policies, Theorem 2 states that as the revenue from the end-product gets smaller, it may be optimal to switch from accepting a demand for the end-product to rejecting

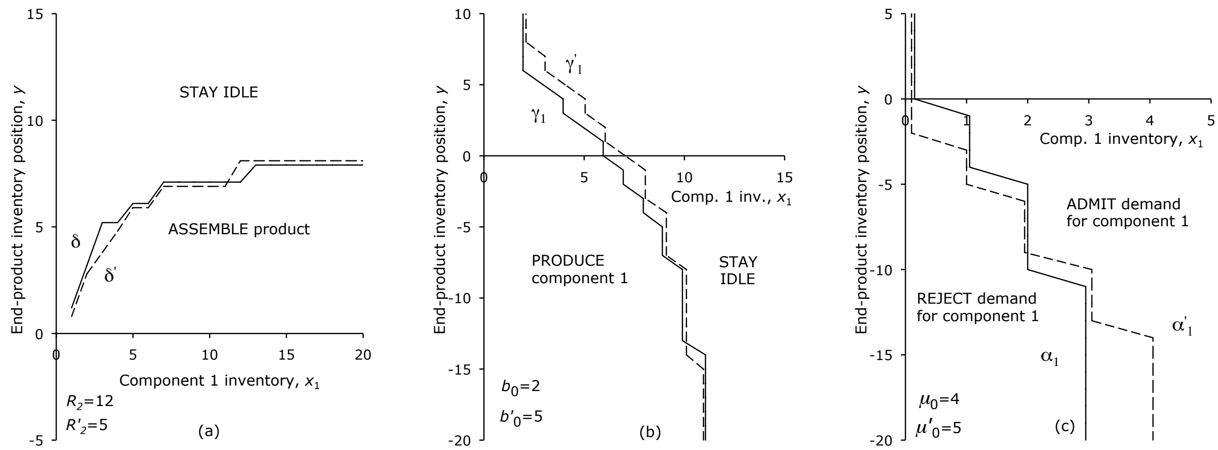

Figure 4: Counter examples for the optimal policy sensitivity on (a) component revenues, (b) backorder cost, and (c) product assembly rate

it, and from rejecting a demand for the intermediate product to accepting it. In terms of the production and assembly policies, it may be optimal to switch from assembling another end-product to staying idle, and from producing a component to staying idle. Figure 3 illustrates the changes in the optimal admission and production/assembly policies due to a product revenue change from $R_0 = 40$ to $R_0' = 25$ with remaining parameters set as $\lambda_0 = 6$, $\lambda_1 = 2$, $\lambda_2 = 3$, $\mu_0 = 8$, $\mu_1 = \mu_2 = 10$, $R_1 = 20$, $R_2 = 10$, $b_0 = 5$, $h_0 = 2$, and $h_1 = h_2 = 1$.

A decrease in the revenue of the end-product reduces the relative importance of satisfying an end-product demand compared to that of satisfying an individual component demand. Therefore, the system tends to admit less end-product demand in the assembly queue, and instead, it accepts more of the demand for the intermediate products. Consequently, this results in fewer end-products to be assembled. For component production, the decreased requirements due to fewer products assembled outweighs the increased requirement due to a higher number of individual component demands satisfied. As a result, fewer components of each type are produced.

Although the optimal policies are monotonic with respect to the end-product revenue, they don't necessarily have uniform monotonicities with respect to other problem parameters. We will show three such cases by way of counter examples. The solid lines in Figure 4 (a)-(c) display the threshold curves for assembly, component production, and component rationing, respectively for a two-component problem with parameters $\lambda_0 = 3, \lambda_1 = \lambda_2 = 2, \mu_0 = 4, \mu_1 = \mu_2 = 6, h_0 = 2, h_1 = h_2 = 1, b_0 = 2, R_0 = 30, R_1 = 10$, and $R_2 = 12$. In each of these figures, the dashed lines refer to the corresponding policies with one of the parameters modified as discussed below.

In Figure 4 (a), we observe the effects of lowering the revenue from a type-2 component on the optimal product assembly threshold. We observe that the threshold curves δ and δ' cross each other, hence the optimal policy does not possess monotonicity with respect to a change in R_2. A low value of R_2 shifts the priority from satisfying individual component type-2 demands towards producing end-products instead. This generates two sorts of dynamics. On the one hand, the priority shift towards the end-product enables the assembly line to have smoother access to component inventories thereby lowers the assembly threshold. On the other hand, as selling individual components separately brings in relatively lower revenues compared to selling them as an end-product, it is more favorable to turn the components into products, thus increasing the assembly threshold. Through numerical studies, we observe the former effect to have a higher influence when component inventories are low to moderate. This makes sense as the competition between the assembly operation and the individual component demands on a component is more critical when component inventories are scarce. At system states with high inventories for both components, the latter effect is more dominant resulting in a higher assembly threshold.

We analyze the effect of increasing the backorder cost on the optimal component production policies in Figure 4 (b). A higher backorder cost requires more of the product demand to be met from inventory and discourages demand admissions to the assembly queue if there are already a high number of backorders in the system. Therefore, for states with on-hand inventories or moderate backorders, we tend to produce more components as we try to meet as much of the product demand as possible without further backordering. However, when there is already a high number of customers waiting in the queue, further product demand admission is prevented, hence the requirement for components decreases.

Finally, in Figure 4 (c) we observe the changes in the optimal component rationing policy based on an increase in the assembly process rate. We again observe two different dynamics. A faster assembly line can make up for a delayed availability of individual components. Therefore it enables more of the individual component demand to be satisfied resulting in lower component rationing thresholds. On the other hand, in order to utilize its fast pace, the assembly operation also requires high component availability allowing quick supplies. We observe that the first effect is stronger at system states where there is on-hand product inventory or only a few customers waiting in the assembly queue. However, when there is a high amount of backorders in the queue, the second effect is influential, saving the components for assembly purposes to quickly lower the number of backorders.

3.2 Extensions to the Model

It is straightforward to extend our model to include multiple customer classes that are willing to pay different amounts for the same end-product as in Benjaafar et al (2010) given that end-products supplied to any customer class require the same processing time and that they have identical backorder costs under which the original state space representation may be retained. In fact, we can also include multiple customer classes for the component demand. For example, let the demand for a type-i component arise by M_i customer classes with arrival rates λ_i^m where $m = 1, 2, ..., M_i$, generating a revenue of R_i^m ranked such that $R_i^1 \geq R_i^2 \geq ... \geq R_i^{M_i}$. In this modified problem, M_i operators of the form $T_i^{1,m} v(\mathbf{x}, y) = \max \left[(v(\mathbf{x} - \mathbf{e}_i, y) + R_i^m) \cdot I_{(x_i > 0)} + v(\mathbf{x}, y) \cdot I_{(x_i = 0)}, v(\mathbf{x}, y) \right]$ replace the original operator T_i^1, resulting in the original problem given by (2) to have the terms $\sum_{m=1}^{M_i} \lambda_i^m \; T_i^{1,m} \; v(\mathbf{x}, y)$ instead of the term $\lambda_i \; T_i^1 \; v(\mathbf{x}, y)$. The conditions set forth in the analysis of Lemma 2 and Theorem 1 suffices to show that the optimal policy has a similar structure as the one described in Theorem 1, except for a replacement of the component demand admission thresholds $\alpha_i(\mathbf{x}_{-i}, y)$ with multiple component admission threshold levels $\alpha_i^1(\mathbf{x}_{-i}, y) \leq \alpha_i^2(\mathbf{x}_{-i}, y) \leq ... \leq \alpha_i^{M_i}(\mathbf{x}_{-i}, y)$.

We also consider another extension to the basic model that takes into account a more general revenue collecting scheme. It is common practice in many businesses that if a customer is to be made to wait for an end-product, only an upfront partial payment for the item is collected rather than the item's full revenue. Consequently, the firm receives the remaining price of the item at the time of delivery. In such a setting, a discounted profit formulation is more valid since we would like to account for the time value of money. Interestingly, the policy structure described in Theorem 1 remains exactly the same with this more general revenue collection scheme.

Let r_0 denote the upfront payment amount received when a customer is admitted to the product assembly queue such that $R_0 \geq r_0 \geq 0$. Further, let the operators corresponding to the product demand admission and product assembly decisions be modified as follows:

$$T_0^1 v(\mathbf{x}, y) = \begin{cases} v(\mathbf{x}, y-1) + R_0 & \text{if } y > 0 \\ \max[v(\mathbf{x}, y-1) + r_0, v(\mathbf{x}, y)] & \text{if } y \leq 0 \end{cases}$$

$$T_0^2 v(\mathbf{x}, y) = \begin{cases} \max \left[(v(\mathbf{x} - \mathbf{1}, y+1)) \cdot I_{(x_i > 0 \; \forall i)} + v(\mathbf{x}, y) \cdot I_{(\exists \; i \; | \; x_i = 0)}, v(\mathbf{x}, y) \right] & \text{if } y \geq 0 \\ \max[(v(\mathbf{x} - \mathbf{1}, y+1) + R_0 - r_0) \cdot I_{(x_i > 0 \; \forall i)} & \\ \qquad + v(\mathbf{x}, y) \cdot I_{(\exists \; i \; | \; x_i = 0)}, v(\mathbf{x}, y)] & \text{if } y < 0 \end{cases}$$

Then, using uniformization with $\Lambda = \phi + \sum_{i=0}^{N} (\lambda_i + \mu_i)$ where ϕ denotes the discount factor, we can rewrite the problem given in (2) as a discounted infinite horizon dynamic program as follows:

$$v(\mathbf{x}, y) = \frac{1}{\Lambda} \left(- \sum_{i=1}^{N} (h_i \; x_i) - h_0 \; y^+ - b_0 \; y^- + \sum_{n=0}^{N} \left(\lambda_j \; T_j^1 \; v(\mathbf{x}, y) + \mu_n \; T_j^2 \; v(\mathbf{x}, y) \right) \right) \tag{3}$$

The following theorem depicts the optimal policy structure.

Theorem 3. *For the problem given in* (3)*, the optimal demand admission, component production and product assembly policies follow the optimal policy structure described in Theorem 1. That is, demand admission for the end-product and the components are characterized by state-dependent admission thresholds and production and assembly decisions follow state-dependent base-stock policies with similar monotonicity properties as set forth in Theorem 1.*

4 A Heuristic Policy

The optimal policy structure determined by Theorem 1 is fairly complex. In addition to the assembly and admission control decisions for the end-product, the firm also needs to make production and rationing decisions for each component. As shown in the previous section, all of these decisions are characterized by state dependent threshold levels. For a general problem with N components, each threshold is defined by a switching surface embedded in an $N+1$ dimensional space. Since the number of possible system states grows exponentially as the number of components gets larger, computing the optimal policy in such cases ceases to be a practical task.

For problems with a limited number of components, however, the optimal switching surfaces may be computed with relative ease as we have previously illustrated in Figure 2. Hence, motivated by the ease of computation for a two-component problem and the prohibitive inefficiencies associated with problems of large sizes, we introduce the following heuristic solution approach. Regarding component production and rationing decisions, for each component type-i, we construct a two-component subproblem P_i that assumes its type-1 component as the component type-i of interest and aggregates all the remaining components into a type-2 component. In the original problem of N components, at each decision epoch corresponding to a demand arrival or production opportunity for component type-i, the heuristic policy maps the system state $(\mathbf{x}, y)$ to state $(x_i, \min\{x_j, j = 1, ..., N, j \neq i\}, y)$ in subproblem P_i and imitates the corresponding decision given at this state for component type-i.

By adopting proxy two-component subproblems, our heuristic policy essentially retains much of the characteristics of the optimal component production and rationing policy structure. For example, the admission decision for a component demand remains a state-dependent rationing policy where this rationing level is nondecreasing with the inventory level of the other component and nonincreasing with the end-product inventory. The underlying assumption that leads us to construct two-component subproblems lies in the intuitive expectation that the production and rationing decisions for a certain component is influenced strongly by the component with the lowest inventory level as opposed to by others with higher inventory levels. In other words, it is the limiting component that mostly impacts the assembly capabilities and hence influences the amount of inventory to hold for others. Although exceptions to this conjecture may occur when there are discrepancies among component production rates, we will maintain this assumption as it allows us to develop a simpler heuristic.

Let the symbol ($\hat{}$) denote the parameters for the constructed two-component subproblem P_i. We define the revenues and cost parameters as $\hat{R_0} = R_0$, $\hat{R_1} = R_i$, $\hat{R_2} = \sum_{j \neq i} R_j$, $\hat{h_1} = h_i$, $\hat{h_2} = \sum_{j \neq i} h_j$, $\hat{h_0} = h_0$, and $\hat{b_0} = b_0$. The demand arrival, production and assembly rates for component type-1 and the end-product in problem P_i are set identically at their values in the original problem. Hence we define $\hat{\lambda_0} = \lambda_0$, $\hat{\lambda_1} = \lambda_i$, $\hat{\mu_0} = \mu_0$, and $\hat{\mu_1} = \mu_i$.

We handle the production and demand arrival rates corresponding to the type-2 component in subproblem P_i rather differently as this component reflects an aggregation of all the remaining ones. Since upon a demand arrival for component type-2, we receive the sum of the revenues from all remaining components, we adjust the demand arrival rate based on the time it takes for a demand to arrive for all components. This rate adjustment calls for a maximum of exponentially distributed random variables where this maximum itself no longer follows exponential distribution. The mean of the maximum of several random variables appears frequently in the reliability literature when calculating the reliability of a system consisting of several servers in parallel. Following equation (7.27) in Billinton and Allan (1983), the mean of the maximum of n independent random variables $(Z_1, \ldots, Z_n)$ where each random variable Z_i is exponentially distributed with mean $1/\lambda_i$ is given by

$$\begin{aligned} E[\max(Z_1, \ldots, Z_n)] = &\left(\tfrac{1}{\lambda_1} + \tfrac{1}{\lambda_2} + \cdots + \tfrac{1}{\lambda_n}\right) - \left(\tfrac{1}{\lambda_1+\lambda_2} + \tfrac{1}{\lambda_1+\lambda_3} + \cdots + \tfrac{1}{\lambda_i+\lambda_j} + \cdots\right) \\ &+ \left(\tfrac{1}{\lambda_1+\lambda_2+\lambda_3} + \tfrac{1}{\lambda_1+\lambda_2+\lambda_4} + \cdots + \tfrac{1}{\lambda_i+\lambda_j+\lambda_k} + \cdots\right) - \cdots + (-1)^{n+1} \tfrac{1}{\sum_{i=1}^{n} \lambda_i} \end{aligned}$$

When calculating the demand arrival rate for the type-2 component, we will treat the inverse of the mean time as the corresponding rate. For the production rate, we set $\hat{\mu_2}$ as the average of the production rate of all the remaining components, i.e., $\hat{\mu_2} = (\sum_{j \neq i} \mu_j)/(N-1)$.

For the component production and rationing decisions, we therefore construct a total of N such two-component subproblems P_i $(i = 1, \ldots, N)$. As for the end-product admission and assembly decisions, we follow an analogous argument by forming a single one-component subproblem P_0, where the parameters for the component are determined by aggregating all the components in a similar fashion. At each decision epoch corresponding to an end-product arrival or assembly opportunity, the heuristic policy maps the system state $(\mathbf{x}, y)$ to state $(\min\{x_i, i = 1, ..., N\}, y)$ in problem P_0 and imitates the corresponding decision given at this state. Likewise, a single component proxy subproblem also possesses much of the optimal policy structure for the end-product, that is, admissions and assembly decisions for the end-product follow state-dependent threshold levels with more units admitted and assembled if the component inventory is higher.

Next, we evaluate the performance of the heuristic policy. Tables 1 and 2 compare the profits obtained by the optimal and the heuristic policies for 24 example problems each for systems consisting of three and four identical components, respectively. For each problem, we report the profits per unit time obtained by the optimal policy, our state-dependent heuristic policy and an independent "strawman" heuristic policy we picked from the literature. As opposed to our heuristic policy described above, the "strawman" heuristic policy is a base-stock/rationing policy for each product where a base-stock and a rationing level is set independently for each product which ignores the inventory levels for all other products. For the three- and four-component problems, we identify the independent base-stock and rationing levels by an exhaustive search on the state space. At each decision epoch corresponding to a production opportunity, the strawman heuristic policy leads to the production of another unit of the product only if its current inventory level is below its independent base-stock level. Similarly, at decision epochs corresponding to demand arrivals, a demand for the product will be accepted only if the product's inventory level is above its independent rationing level. Independent base-stock and rationing heuristic policies are commonly used in the related literature such as in the works of Song et al. (1999) and Benjaafar et al. (2010).

For each problem instance, the profits reported for the optimal policy is obtained by solving the corresponding MDP by value iteration algorithm with a five-digit accuracy termination criteria. For our heuristic policy, we first solve the $N + 1$ constructed MDPs, recording the decisions for each component and product, and then apply these decisions for the original-sized MDP. For the independent base-stock and rationing policy, we first identify the appropriate base-stock and admission threshold levels for each component and product which are then implemented in the value iteration algorithm.

In Tables 1 and 2, the parameters varied include the revenues from the intermediate and end-products, the demand arrival rates for the intermediate and end-products, and the utilizations for the production and assembly lines. The revenue parameters are selected to allow the testing of the heuristics for cases where the revenue from an end-product is higher, equal to, or lower than the sum of the revenues from intermediate components. Specifically, for both the three- and four-component problems, we evaluate the heuristics where the end-product revenue is $\frac{2}{3}, 1, \frac{4}{3}$ times the sum of the individual component revenues. For example, for the case of three intermediate components, the price pair $R_0 = 30$ and $R_i = 15$ corresponds to a revenue ratio of $\frac{2}{3}$ while the price

Table 1: Performance of the heuristics and independent base-stock/rationing policy for a system with three identical intermediate products

No	R_0	R_i	λ_0	λ_i	ρ_0	ρ_i	Optimal Profit	Heuristic Profit	Heuristic % diff.	Indep. BS&R Profit	Indep. BS&R % diff.
1	30	10	4	3	0.5	0.5	193.2	193.1	0.02	191.4	0.91
2	30	10	4	3	0.5	0.9	178.6	177.4	0.68	175.2	1.89
3	30	10	4	3	0.9	0.5	178.9	178.9	0.01	177.0	1.03
4	30	10	4	3	0.9	0.9	167.0	166.6	0.24	164.4	1.56
5	30	10	6	0.5	0.5	0.5	181.2	181.2	0.01	178.2	1.70
6	30	10	6	0.5	0.5	0.9	162.0	161.3	0.46	158.6	2.10
7	30	10	6	0.5	0.9	0.5	163.2	163.1	0.02	159.8	2.04
8	30	10	6	0.5	0.9	0.9	150.5	150.3	0.11	148.4	1.38
9	30	15	4	3	0.5	0.5	232.7	232.2	0.19	230.1	1.08
10	30	15	4	3	0.5	0.9	213.1	209.9	1.51	208.7	2.08
11	30	15	4	3	0.9	0.5	217.7	217.6	0.07	216.4	0.61
12	30	15	4	3	0.9	0.9	201.6	200.9	0.36	199.3	1.15
13	30	15	6	0.5	0.5	0.5	185.3	185.3	0.04	172.2	7.08
14	30	15	6	0.5	0.5	0.9	163.3	161.9	0.87	158.8	2.81
15	30	15	6	0.5	0.9	0.5	165.9	165.8	0.03	162.4	2.08
16	30	15	6	0.5	0.9	0.9	150.4	150.0	0.25	148.4	1.35
17	60	15	4	3	0.5	0.5	346.9	346.9	0.00	344.1	0.81
18	60	15	4	3	0.5	0.9	323.7	323.0	0.21	316.4	2.25
19	60	15	4	3	0.9	0.5	319.2	319.1	0.02	316.9	0.72
20	60	15	4	3	0.9	0.9	300.2	299.9	0.09	295.1	1.70
21	60	15	6	0.5	0.5	0.5	358.5	358.5	0.00	353.1	1.51
22	60	15	6	0.5	0.5	0.9	324.6	323.7	0.28	321.8	0.87
23	60	15	6	0.5	0.9	0.5	323.9	323.9	0.00	319.1	1.50
24	60	15	6	0.5	0.9	0.9	302.0	301.8	0.09	297.6	1.47

pair $R_0 = 60$ and $R_i = 15$ corresponds to a revenue ratio of $\frac{4}{3}$. Generally, due to further processing, the end-product revenue may be at least as much as the sum of the revenues of its constituents. However, there may be examples where the reverse holds, such as after-sales parts that are sold at much higher prices. The revenue ratio of $\frac{2}{3}$ enables us to investigate the performance of the heuristics in such settings. We also change the intermediate and end-product demand arrival rates between a high and a low ratio to observe the cases where individual sales are a significant part of the business and the cases where the focus is overwhelmingly on the end-product with occasional demands arising for intermediate products. Finally, we vary both component production and assembly utilizations between low and high values by selecting μ_0 and μ_i such that $\frac{\lambda_0}{\mu_0} = \rho_0$ and $\frac{\lambda_i+\lambda_0}{\mu_i} = \rho_i \;\; \forall\, i = 1, ..., 4$. Throughout these example problems, we assume $b_0 = 0.2R_0$, $h_0 = 0.1R_0$, and $h_i = 0.1R_i \; \forall\, i = 1, \ldots, N$.

In Table 1, the average difference between the profits obtained by the optimal and the heuristic policy is 0.23% whereas the profit difference between the optimal policy and the independent base-stock/rationing policy is 1.74%. By making use of the inventory positions of the end-product and the limiting component, the heuristic policy performs better than the independent base-stock/rationing policy which only uses local inventory information. In fact, we observe that the heuristic policy-outperforms the independent base-stock/rationing policy in all instances of the example problems.

Regarding product revenues, we find that the profit attained by the heuristic policy differs from that of the optimal policy for an average value of $0.41, 0.19$ and 0.09 corresponding to product revenue ratios of $\frac{2}{3}, 1$, and $\frac{4}{3}$, respectively, i.e. averages from problems No. 9-16, 1-8 and 17-24. Thus, we observe that the performance of the heuristic policy improves as the revenue from the end-product increases with respect to the sum of the revenues from intermediate components. In terms of demand arrival rate ratios, we find that the heuristic policy performs slightly better at high arrival rates for the end-product and low arrival rates for intermediate products. These two properties suggest that the heuristic policy is also capable of controlling pure assembly systems with no exogenous demand for the intermediate components. Lastly, we observe that the heuristic policy performs very well for problem instances where utilization for the production line is low with an average difference of 0.04% between the profits obtained by the optimal and the heuristic. This is somewhat expected since the heuristic plans the production of an item based on the inventory of the limiting product. A faster production rate allows withholding the processing of an item until the inventory position of the limiting item is restored. In settings where the production line utilization is high and the assembly line utilization is low, however, we find that the performance of the heuristic policy is degraded to an average difference of 0.67%. On the other hand, settings with low assembly utilizations also lead to even lower performance by the independent policy with an average difference of 2.09% and a maximum difference of up to 7.08% as observed in problem No. 13.

Table 2 is constructed in a similar fashion to Table 1 in order to evaluate the performance of the heuristic policy when applied to systems with a larger number of intermediate products. We find that the average difference between the optimal and the heuristic profit is 0.31% while the average difference between the optimal profit and the profit obtained by the independent base-stock/rationing policy is 1.91%. Comparing the three- and four-component results, an important characteristic of the heuristic policy seems to be its retained robustness moving from a three-component problem to a four-component one. In accordance with the results obtained by Table 1, a closer look into Table 2 also reveals that the performance of the heuristic policy is strongest in settings where the revenue from the end-product is higher compared to the sum of the revenues from intermediate components, when the demand rate for the end-product is higher compared to the demand rate for intermediate components, and when the production line utilizations are low. The heuristic is robust with respect to the assembly line utilization.

Next, we test how the heuristic policy performs in systems with asymmetric rate, revenue and

Table 2: Performance of the heuristics and an independent base-stock/rationing policy for a system with four identical intermediate products

No	R_0	R_i	λ_0	λ_i	ρ_0	ρ_i	Optimal Profit	Heuristic Profit	Heuristic % diff.	Indep. BS&R Profit	Indep. BS&R % diff.
1	40	10	4	3	0.5	0.5	257.1	257.0	0.07	254.8	0.91
2	40	10	4	3	0.5	0.9	236.9	234.7	0.90	232.2	1.97
3	40	10	4	3	0.9	0.5	238.0	237.9	0.04	235.6	1.02
4	40	10	4	3	0.9	0.9	221.5	220.7	0.35	216.3	2.34
5	40	10	6	0.5	0.5	0.5	240.9	240.8	0.05	235.7	2.17
6	40	10	6	0.5	0.5	0.9	213.1	211.7	0.67	208.6	2.13
7	40	10	6	0.5	0.9	0.5	216.8	216.7	0.00	212.4	2.04
8	40	10	6	0.5	0.9	0.9	198.2	198.0	0.07	195.3	1.45
9	40	15	4	3	0.5	0.5	309.6	309.2	0.14	305.0	1.49
10	40	15	4	3	0.5	0.9	282.3	277.1	1.82	272.4	3.50
11	40	15	4	3	0.9	0.5	289.5	289.4	0.05	287.2	0.79
12	40	15	4	3	0.9	0.9	267.0	266.1	0.36	264.1	1.11
13	40	15	6	0.5	0.5	0.5	246.2	245.9	0.11	227.1	7.76
14	40	15	6	0.5	0.5	0.9	214.0	211.2	1.33	208.1	2.76
15	40	15	6	0.5	0.9	0.5	220.0	219.9	0.06	215.7	1.96
16	40	15	6	0.5	0.9	0.9	197.4	196.8	0.33	194.1	1.69
17	80	15	4	3	0.5	0.5	461.9	461.7	0.05	457.9	0.86
18	80	15	4	3	0.5	0.9	429.7	428.4	0.32	417.8	2.79
19	80	15	4	3	0.9	0.5	424.8	424.6	0.05	421.5	0.76
20	80	15	4	3	0.9	0.9	398.6	398.0	0.15	393.7	1.22
21	80	15	6	0.5	0.5	0.5	476.9	476.8	0.02	471.8	1.05
22	80	15	6	0.5	0.5	0.9	428.0	426.2	0.41	425.1	0.67
23	80	15	6	0.5	0.9	0.5	430.7	430.6	0.01	422.7	1.85
24	80	15	6	0.5	0.9	0.9	399.1	398.4	0.16	393.1	1.49

cost parameters. In Table 3, we construct a base case labeled as problem No. 0, for which we set $R_0 = 50, R_1 = 20, R_2 = 15, R_3 = 10, R_4 = 5$ for revenues, $\lambda_0 = 10, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \lambda_4 = 4$ for demand arrival rates, $\rho_0 = 0.7, \rho_1 = 0.8, \rho_2 = 0.9, \rho_3 = 0.5, \rho_4 = 0.6$ for utilizations, and $b_0 = 0.2R_0, h_0 = 0.1R_0, R_i = 0.1 \forall i$ for backorder and holding cost parameters. In the 18 instances to follow, a parameter corresponding to the end-product and one of the intermediate products (product type-1) is either increased or decreased. For this experiment, we keep the parameters for the remaining intermediate products unchanged and hence omit representing their values in Table 3. In example 1, the revenue from the end-product is much higher than the sum of the revenues from intermediate products. In example 2, on the other hand, the end-product revenue is lower. Similarly, examples 3 and 4 depict instances when the revenue from the intermediate product type-1 is high and low. Examples 5-8 correspond to high and low demand arrival rates for the end-product and the intermediate product. Examples 9-12 explore the effects of high and low assembly and production utilizations. Finally, in examples 13-15 we change the backorder and holding costs for the end-product and the intermediate product type-1.

In Table 3, we observe that the difference between the profits obtained by the optimal and the

Table 3: Performance of the heuristics and an independent base-stock/rationing policy for an asymmetric system with four intermediate products

No	R_0	R_1	λ_0	λ_1	ρ_0	ρ_1	b_0	h_0	h_1	Optimal Profit	Heuristic Profit	Heuristic % diff.	Indep. BS&R Profit	Indep. BS&R % diff.
0	50	20	10	1	0.7	0.8	10	5	2	552.1	551.0	0.20	540.8	2.05
1	100	20	10	1	0.7	0.8	10	5	2	1058.2	1057.7	0.05	1044.9	1.26
2	30	20	10	1	0.7	0.8	10	5	2	356.5	349.8	1.88	345.8	3.00
3	50	40	10	1	0.7	0.8	10	5	2	571.4	570.3	0.19	561.4	1.75
4	50	5	10	1	0.7	0.8	10	5	2	538.8	537.8	0.17	526.8	2.22
5	50	20	20	1	0.7	0.8	10	5	2	1046.6	1045.9	0.07	1035.2	1.09
6	50	20	5	1	0.7	0.8	10	5	2	307.0	306.0	0.32	299.0	2.62
7	50	20	10	5	0.7	0.8	10	5	2	630.5	629.6	0.13	612.4	2.87
8	50	20	10	0.2	0.7	0.8	10	5	2	536.3	535.3	0.17	527.5	1.64
9	50	20	10	1	0.9	0.8	10	5	2	528.3	527.7	0.15	522.2	1.15
10	50	20	10	1	0.5	0.8	10	5	2	560.6	560.0	0.11	532.1	5.09
11	50	20	10	1	0.7	0.9	10	5	2	545.5	543.0	0.47	537.0	1.56
12	50	20	10	1	0.7	0.4	10	5	2	559.7	559.2	0.09	548.9	1.93
13	50	20	10	1	0.7	0.8	20	5	2	544.5	542.8	0.31	535.7	1.61
14	50	20	10	1	0.7	0.8	5	5	2	560.3	559.7	0.11	552.1	1.47
15	50	20	10	1	0.7	0.8	10	10	2	545.1	544.5	0.11	539.6	1.01
16	50	20	10	1	0.7	0.8	10	2	2	562.9	561.5	0.26	554.8	1.45
17	50	20	10	1	0.7	0.8	10	5	4	545.0	543.6	0.26	526.5	3.39
18	50	20	10	1	0.7	0.8	10	5	1	557.2	556.4	0.14	550.3	1.24

heuristic policy is 0.27% whereas the difference between the profits attained by the optimal and the independent base-stock/rationing policy is 2.02%. Hence, we find that the heuristic policy maintains its performance when there are differences in the rate, revenue, and cost parameters of various intermediate products. In addition, as was the case in Tables 1 and 2, the heursitic policy performs better than the independent base-stock/rationing policy in every problem instance. Through the experiment in Table 3, we observe that the performance of the heuristic policy improves when the end-product revenue and demand rate is high and when production line utilization is low. The heuristic also performed better when assembly backorder cost was low, end-product holding cost was high, and intermediate product holding cost was low. The heuristic has been robust with respect to changes in the revenue and demand rate of the intermediate component as well as the assembly utilization.

Contemplating the performance of our heuristic policy against the optimal policy across Tables 1-3, we observe that the largest deviation of profits between the heuristics and the optimal policy occurs at instances when the end-product revenue is low compared to the revenue from individual products and when production line utilization is high (e.g., Instance No. 10 in Tables 1 and 2, and instances No. 2 and No. 11 in Table 3). As we have shown in Theorem 2, the firm assembles fewer end-products when the end-product revenue is low. In addition, high utilization in production lines impedes the firms ability to coordinate and delay production of an item with respect to

Table 4: Performance of the heuristics and an independent base-stock/rationing policy for an asymmetric system with six intermediate products

No	R_0	R_1	λ_0	λ_1	ρ_0	ρ_1	Heuristic Profit (95% C.I.)	Indep. BS&R Profit (95% C.I.)	% diff.
0	150	75	10	1	0.7	0.8	1574.4 (± 6.6)	1526.5 (± 5.1)	3.04
1	200	75	10	1	0.7	0.8	2057.2 (± 5.9)	1987.2 (± 6.7)	3.40
2	100	75	10	1	0.7	0.8	1099.3 (± 4.1)	1058.0 (± 4.8)	3.76
3	150	100	10	1	0.7	0.8	1581.9 (± 6.4)	1525.4 (± 6.0)	3.57
4	150	25	10	1	0.7	0.8	1552.8 (± 5.8)	1529.6 (± 6.8)	1.49
5	100	75	20	1	0.7	0.8	3052.2 (± 9.5)	3010.2 (± 11.5)	1.38
6	150	75	5	1	0.7	0.8	839.5 (± 3.4)	765.8 (± 5.3)	8.78
7	150	75	10	4	0.7	0.8	1786.7 (± 5.5)	1749,6 (± 8.4)	2.08
8	150	75	10	0.2	0.7	0.8	1510.9 (± 6.0)	1459.8 (± 5.3)	3.38
9	150	75	10	1	0.9	0.8	1493.5 (± 6.2)	1440.2 (± 6.1)	3.57
10	150	75	10	1	0.5	0.8	1602.1 (± 6.4)	1569.5 (± 6.2)	2.03
11	150	75	10	1	0.7	0.9	1516.2 (± 6.0)	1430.0 (± 6.0)	5.69
12	150	75	10	1	0.7	0.6	1625.8 (± 7.2)	1604.8 (± 6.9)	1.29

the inventory positions of other items. These two characteristics diminishes the importance of coordination among component production and inhibit the advantages of the heuristic policy which is designed to coordinate production of components based on limiting inventories.

We are also interested to see the performance of the heuristic policy for even larger systems. Although we can compute the decisions for the heuristic policy with ease for an arbitrarily large system, it is not possible to evaluate the profits from the optimal and heuristic policies by solving MDPs since the computational load grows exponentially with the number of products. Therefore, we resort to a simulation approach to test the performance of our heuristic policy against the independent strawman base-stock and rationing heuristic policy. For a large system, finding the appropriate base-stock and rationing parameters for the strawman heuristic policy is a strenuous task. Hence, we adapt the following simpler procedure. To find the base-stock and rationing levels for component type-i, we form a two customer class, single server system that produces to stock. The first customer class is the original demand process for the individual component type-i whereas the second customer class represents the end-product bringing in a revenue of $R_0(R_i/\sum R_i)$ with an arrival rate of λ_0. Utilizing this model, we can easily identify both the base-stock and rationing level for component type-i which serves as our estimate for the independent base-stock and rationing level. For the assembly process, we form a single server, single demand class system that allows backorders in order to identify an appropriate base stock and admission threshold level for the end-product.

Table 4 summarizes the results from various instances of a six-component problem. The base case, labeled as problem No. 0 corresponds to a problem where the revenues are set as $R_0 = 150, R_1 = 75, R_2 = 30, R_3 = 25, R_4 = 10, R_5 = R_6 = 5$, the arrival rates are $\lambda_0 = 10, \lambda_1 =$

$1, \lambda_2 = \lambda_3 = 2, \lambda_4 = \lambda_5 = \lambda_6 = 3$, utilizations are $\rho_0 = 0.7, \rho_i = 0.8 \forall i$, and backorder and holding costs are $b_0 = 0.2R_0, h_0 = 0.1R_0, h_i = 0.1R_i \forall i$. In the next 12 instances, a parameter corresponding to the end-product and one of the intermediate products (product type-1) is either increased or decreased, similar to the instances studied in Table 3. For each of the problem instances, we run 25 simulations of 80,000 events (e.g., demand arrivals, production opportunities) each to calculate the average profit obtained by our heuristic policy and the independent base-stock and rationing heuristic policy. We report the mean value for the average profits as well as their 95% confidence intervals. In each of the instances, we observe that our heuristic policy outperforms the independent base-stock and rationing heuristic policy with non-overlapping confidence intervals. The average difference between the profits obtained by our heuristic policy and the independent base-stock/rationing policy is 3.34%.

We would like to end this section with a note on systems with a larger number of components. The computational requirements for the heuristic policy grow linearly with the number of components (a total of $N + 1$ subproblems need to be constructed and solved optimally for a system with N components). As the results in Tables 1-4 corresponding to three to six-component problems imply, we expect the heuristic to maintain a highly satisfactory level of performance for systems with a moderate large number of components. Therefore, due to its performance in the problems tested, ease of implementation, and requirement of only a manageable number of subproblems, we believe this heuristic policy would be very effective and beneficial for the control of such systems in practice.

5 Conclusions

In this paper, we studied production and demand admission decisions in an assembly system where there is demand for both the end-product and intermediate products. For a general system composed of an arbitrary number of components, we showed that demand admission for the end-product and for any of the intermediate products are characterized by state-dependent rationing and admission threshold levels while both component production and product assembly follow state-dependent base-stock levels. We explored the sensitivity properties of the optimal policy to various problem parameters. In addition, we provided two extensions for the basic model, one concerning with multiple customer classes based on revenue in addition to the classes based on the type of item they request, and the other, investigating the effects of a partial payment scheme on the optimal policy structure. Since the optimal policies were rather complex and defined by switching surfaces in a multidimensional space, we also introduced a heuristic policy that performed well under a variety of example problems. Extensions that allow customer demands for a selection of components may constitute interesting and challenging problems for future research. The scope of this paper may be regarded as a special case of the general assemble-to-order problem for which assembly is done from a selection of components chosen by a customer. The optimal policies have not been fully characterized for such systems and extensions to this research may provide valuable additional results and insights applicable to these problems.

Acknowledgements

The authors would like to express gratitude for the support of the National Science Foundation Engineering Research Center for Reconfigurable Manufacturing Systems at the University of Michigan (NSF Grant No. EEC-9529125) for providing funding for this research. We also thank the area editor and two anonymous referees for their helpful comments that have improved the paper.

6 Appendix

Proof of Lemma 1: For brevity, we only show the proof for (iv) as the analysis for the remaining conditions are similar. $D_1 v(\mathbf{x}, y) \downarrow x_i$: $D_{\mathbf{1}} v(\mathbf{x} + \mathbf{e}_i, y) - D_{\mathbf{1}} v(\mathbf{x}, y) = v(\mathbf{x} + \mathbf{e}_i + \mathbf{1}, y) - v(\mathbf{x} + \mathbf{e}_i, y) - v(\mathbf{x} + \mathbf{1}, y) + v(\mathbf{x}, y) \leq v(\mathbf{x} + \mathbf{e}_i + \mathbf{1}, y) - v(\mathbf{x} + \mathbf{e}_i, y+1) - v(\mathbf{x} + \mathbf{1}, y) + v(\mathbf{x}, y+1) = -D_{\mathbf{-1},p} v(\mathbf{x} + \mathbf{e}_i, y) + D_{\mathbf{-1},p} v(\mathbf{x}, y) \leq 0$ where the first and second inequalities follow from (i) and (iii), respectively. $D_1 v(\mathbf{x}, y) \downarrow y$: By expanding and regrouping the terms we get $D_{\mathbf{1}} v(\mathbf{x}, y+1) - D_{\mathbf{1}} v(\mathbf{x}, y) = D_p v(\mathbf{x}+\mathbf{1}, y) - D_p v(\mathbf{x}, y) \leq 0$ where the inequality follows from successive applications of $D_p \downarrow x_i$ in (ii).

Proof of Lemma 2: For brevity, we only present the proofs for the preservation of the supermodularity condition given in (i), i.e. $v(\mathbf{x}, y)$ is supermodular in (x_i, x_j), and the submodularity condition given in (iii), i.e. $D_{\mathbf{-1},p} v(\mathbf{x}, y) \downarrow y$, as the proofs corresponding to the remaining conditions are similar. Topkis (1998) shows that maximizing a function that is supermodular in its action and state arguments retains this property. In our setting however, the operators yield functions that are supermodular in some action and states while simultaneously exhibiting submodularity in others. Although this prevents us from using the results in Topkis (1998) directly, we follow the approach therein and use the framework given in works such as Ha (1997a), Carr and Duenyas (2000), Benjaafar et al (2010).

To prove $v(\mathbf{x}, y)$ is supermodular in (x_i, x_j), we show that if v satisfies $D_i v(\mathbf{x}, y) \uparrow x_j$, then $T_0^1 v$, $T_i^1 v$, $T_j^1 v$, $T_k^1 v$, $T_0^2 v$, $T_i^2 v$, $T_j^2 v$, $T_k^2 v$, and Tv (where i, j, k are distinct) all satisfy the same condition. Koole (2006) provides propagation results for a rich set of properties and operators that are frequently encountered in the analysis of production systems including single server and tandem settings. The propagation results for the operators $T^1_0 v$, $T^1_i v$, $T^1_j v$, $T^1_k v$, $T^2_i v$, $T^2_j v$, and $T^2_k v$ follow similarly from Definitions 5.2 and 6.3, and Theorem 7.2 in Koole (2006). We show the results for the operator $T^2_0 v$ corresponding to the end-product assembly decisions which does not follow immediately from the tandem operators considered in Koole (2006).

For $x_i > 0\ \forall i$, we have

$$D_i T^2_0 v(\mathbf{x}+\mathbf{e}_j, y) = \max[v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j-\mathbf{1}, y+1), v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j, y)] - \max[v(\mathbf{x}+\mathbf{e}_j-\mathbf{1}, y+1), v(\mathbf{x}+\mathbf{e}_j, y)] \quad (4)$$

$$D_i T^2_0 v(\mathbf{x}, y) = \max[v(\mathbf{x}+\mathbf{e}_i-\mathbf{1}, y+1), v(\mathbf{x}+\mathbf{e}_i, y)] - \max[v(\mathbf{x}-\mathbf{1}, y+1), v(\mathbf{x}, y)] \quad (5)$$

Eliminating the infeasible outcomes due to (iii), the result for (4) is either $D_i v(\mathbf{x}+\mathbf{e}_j-\mathbf{1}, y+1)$, $v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j-\mathbf{1}, y+1) - v(\mathbf{x}+\mathbf{e}_j, y)$, or $D_i v(\mathbf{x}+\mathbf{e}_j, y)$ and simple algebra shows that (4)-(5) ≥ 0 in all three cases and the corresponding outcomes of (5). For boundary states where $x_i = 0$, $x_j = 0$, or $x_k = 0$, the analysis for the feasible cases are similar.

The result for the operator T follows immediately. By definition, T is formed by (a) the addition and multiplication of positive constants with the functions $T^1_n v$ and $T^2_n v$ for $n = 0,...,N$ that are shown to be $\uparrow x_j$ and (b) linear inventory holding and assembly queue backorder costs. Therefore, $D_i T v \uparrow x_j$ as well.

Next, we show $D_{\text{-}\mathbf{1},p} v(\mathbf{x}, y) \downarrow y$. We only present $D_{\text{-}\mathbf{1},p} T^1_0 v(\mathbf{x}, y) \downarrow y$ as the analysis for the remaining operators $T^1_i v$, $T^2_0 v$, $T^2_i v$, and Tv are similar. For $y > 0$, as we satisfy the demand from available inventory, $D_{\text{-}\mathbf{1},p} T^1_0 v(\mathbf{x}, y+1) - D_{\text{-}\mathbf{1},p} T^1_0 v(\mathbf{x}, y) = D_{\text{-}\mathbf{1},p} v(\mathbf{x}, y) - D_{\text{-}\mathbf{1},p} v(\mathbf{x}, y-1) \leq 0$ by $D_{\text{-}\mathbf{1},p} v \downarrow y$. For $y \leq 0$, we have

$$D_{\text{-}\mathbf{1},p} T^1_0 v(\mathbf{x}, y+1) = \max[v(\mathbf{x}-\mathbf{1}, y+1), v(\mathbf{x}-\mathbf{1}, y+2)] - \max[v(\mathbf{x}, y), v(\mathbf{x}, y+1)] \quad (6)$$

$$D_{\text{-}\mathbf{1},p} T^1_0 v(\mathbf{x}, y) = \max[v(\mathbf{x}-\mathbf{1}, y), v(\mathbf{x}-\mathbf{1}, y+1)] - \max[v(\mathbf{x}, y-1), v(\mathbf{x}, y)] \quad (7)$$

We need to show that (6) minus (7) ≤ 0. The only feasible outcomes for (6) are $D_{\text{-}\mathbf{1},p} v(\mathbf{x}, y+1)$, $v(\mathbf{x}-\mathbf{1}, y+1) - v(\mathbf{x}, y+1)$, and $D_{\text{-}\mathbf{1},p} v(\mathbf{x}, y)$ for which simple algebra yields (6) minus (7) ≤ 0 for all three cases and the corresponding outcomes of (7). □

Proof of Theorem 1: Consider a value iteration algorithm to solve the optimal policy for the problem given in (2) where initial values $v_0(\mathbf{x}, y) = 0$ are used for every state $(\mathbf{x}, y)$. Conditions (i)-(iii) are trivially satisfied by $v_0(\mathbf{x}, y)$, hence $v_0(\mathbf{x}, y) \in V$. We apply $v_{k+1}(\mathbf{x}, y) = T v_k(\mathbf{x}, y)$ for $k = 0, 1, 2, ...$ to determine the relative value functions for successive iterations. Suppose now that the value functions in iteration k satisfy (i)-(iii), i.e. $v_k(\mathbf{x}, y) \in V$. Then, Lemma 1 shows that $v_{k+1}(\mathbf{x}, y)$ also satisfy (i)-(iii). Therefore $v_{k+1}(\mathbf{x}, y) \in V$.

We note that, without loss of optimality, we can add the following constraints to the original problem that we cannot admit a product demand when $R_0 < b_0\ y^-/\Lambda$, we cannot produce a component type-i when $h_i\ x_i/\Lambda > \max(R_i, (b_0 + \sum_{i=1}^{N} h_i/\Lambda))$, and we cannot assemble another end product when $h_0\ y^+/\Lambda > \max(R_0, ((\sum_{i=1}^{N} h_i) - h_0)/\Lambda)$. For example, if $b_0\ y^-/\Lambda > R_0$, this suggests that the amount of backorder cost incurred until the next transition is greater than any potential revenue of R_0 that would be received if the next event were a product demand arrival. (As another example, if $h_i\ x_i/\Lambda > \max(R_i, (b_0 + \sum_{i=1}^{N} h_i/\Lambda))$, this indicates that the amount of holding cost due to a type-i component incurred during a transition epoch is greater than the potential benefits of (a) selling that component for a revenue of R_i were the next event a demand arrival for component i, and (b) assembling another unit of a backordered product that would save backorder and holding costs for a transition epoch if the next event were a product assembly opportunity.) Thus, the original problem can be converted to a finite state, finite action set problem. The underlying Markov chain is also unichain. Thus, Theorem 8.4.5 of Puterman (1994) ensures the existence of a long-run average profit and the validity of the value iteration algorithm to determine it.

To complete the proof of Theorem 1, we note that conditions (i)-(iii) are sufficient to demonstrate the structural properties of the optimal policy. Due to (i), if it is optimal not to produce component i in state $(\mathbf{x}, y)$, it remains optimal not to do so in state $(\mathbf{x} + \mathbf{e}_i, y)$, implying a base-stock production policy. Further, the sub- and super-modularity conditions imply that the base-stock level is nondecreasing with the inventory of other components and nonincreasing with the end-product inventory. Condition (i) also implies that if it is optimal to accept a demand for component i in state $v(\mathbf{x}, y)$, i.e., $R_i \geq v(\mathbf{x}, y) - v(\mathbf{x} - \mathbf{e}_i, y)$, it is also optimal to accept a demand for component i in state $v(\mathbf{x} + \mathbf{e}_i, y)$, i.e. $R_i \geq v(\mathbf{x} + \mathbf{e}_i, y) - v(\mathbf{x}, y)$. Thus component demand admission follows a rationing policy. Similarly, the sub- and super-modularity conditions imply that the rationing level for a component is nondecreasing with the inventory of other products and nonincreasing with the end-product inventory position.

Condition (ii) indicates that if $v(\mathbf{x}, y - 1) + R_0 \geq v(\mathbf{x}, y)$, then $v(\mathbf{x}, y) + R_0 \geq v(\mathbf{x}, y + 1)$, thus implying an admission threshold for the end-product. Moreover, $D_{-P} \uparrow x_i$ implies that the state dependent admission threshold is nondecreasing with the amount of component inventories. Finally, condition (iii) implies the structure of the optimal product assembly policy. $D_{\text{-}\mathbf{1},p} \downarrow y$ means that the additional value gained by assembling an end-product gets smaller with each additional unit of end-product in the inventory, implying that product assembly follows a state dependent threshold structure. $D_{\text{-}\mathbf{1},p} \uparrow x_i$ suggests that the assembly threshold level is non-decreasing with the amount of component inventories. $\square$

Proof of Theorem 2: We construct two systems that are identical in all problem parameters except the end-product revenues which are chosen such that $R_0' < R_0$. We refer to the original problem where the end product revenue is R_0 as problem A, and the modified setting with R_0' as problem B. We initialize problems A and B with $v_0(\mathbf{x}, y) = 0$ and $v_0'(\mathbf{x}, y) = 0$, respectively. We then apply $v_{k+1}(\mathbf{x}, y) = Tv_k(\mathbf{x}, y)$ and $v_{k+1}'(\mathbf{x}, y) = Tv_k'(\mathbf{x}, y)$. By Lemma 2, $v_k(\mathbf{x}, y)$ and $v_k'(\mathbf{x}, y)$ satisfy conditions (i)-(iii) $\forall$ k. Hence both v_k and $v_k' \in V$. We first prove the following lemma.

Lemma 3. *Let v_k and $v_k' \in V$ for $\forall$ k. For each state $(\boldsymbol{x}, y)$ and for every $k = 0, 1, 2, \ldots$, the following conditions jointly hold:*

(a) $D_i v_k'(\mathbf{x}, y) - D_i v_k(\mathbf{x}, y) \leq 0$

(b) $D_p v_k'(\mathbf{x}, y) - D_p v_k(\mathbf{x}, y) + R_0 - R_0' \geq 0$

(c) $D_{\mathbf{-1},p} v_k'(\mathbf{x}, y) - D_{\mathbf{-1},p} v_k(\mathbf{x}, y) \leq 0$

Proof: The proof of Lemma 3 is by induction. Conditions (a)-(c) hold trivially for $v_0(\mathbf{x}, y)$ and $v_0'(\mathbf{x}, y)$. We assume the conditions hold for iteration k and show that they are preserved in iteration $k+1$. For brevity, we only present the proof that $D_p v_k'(\mathbf{x}, y) - D_p v_k(\mathbf{x}, y) + R_0 - R_0' \geq 0$. The proofs of other conditions are similar and hence omitted. Since $v_{k+1}(\mathbf{x}, y) = Tv_k(\mathbf{x}, y)$ and $v_{k+1}'(\mathbf{x}, y) = Tv_k'(\mathbf{x}, y)$, we have

$$D_p v_{k+1}'(\mathbf{x}, y) - D_p v_{k+1}(\mathbf{x}, y) = D_p T v_k'(\mathbf{x}, y) - D_p T v_k(\mathbf{x}, y)$$

As in the proof of Lemma 2, we start by showing $D_p T_0^1 v_k'(\mathbf{x}, y) - D_p T_0^1 v_k(\mathbf{x}, y) + R_0 - R_0' \geq 0$ and proceed to show that the condition holds for the remaining operators T_i^1, T_0^2, T_i^2, and T.

$$\begin{aligned} &D_p T_0^1 v_k'(\mathbf{x}, y) - D_p T_0^1 v_k(\mathbf{x}, y) + R_0 - R_0' \\ &\qquad = \max[v_k'(\mathbf{x}, y) + R_0', v_k'(\mathbf{x}, y+1)] - \max[v_k'(\mathbf{x}, y-1) + R_0', v_k'(\mathbf{x}, y)] \\ &\qquad\quad - \max[v_k(\mathbf{x}, y) + R_0, v_k(\mathbf{x}, y+1)] + \max[v_k(\mathbf{x}, y-1) + R_0, v_k(\mathbf{x}, y)] + R_0 - R_0' \end{aligned} \tag{8}$$

In order to simplify the analysis, we adapt a similar notation used in Carr and Duenyas (2000) and introduce two functions w and w' on $\{0, 1\}$ x S. Let w be defined as

$$w(u, \mathbf{x}, y) = \begin{cases} v_k(\mathbf{x}, y-1) + R_0 & \text{if } u = 1 \\ v_k(\mathbf{x}, y) & \text{if } u = 0 \end{cases}$$

and w' be defined similarly with the corresponding value function v_k' and product revenue R_0'. Therefore $T_0^1 v_k(\mathbf{x}, y) = \max_{u \in 0,1} w(u, \mathbf{x}, y)$ and $T_0^1 v_k'(\mathbf{x}, y) = \max_{u \in 0,1} w'(u, \mathbf{x}, y)$. We let $u_{(\mathbf{x},y)} = \text{argmax}_u w(u, \mathbf{x}, y)$ and $u_{(\mathbf{x},y)}' = \text{argmax}_u w'(u, \mathbf{x}, y)$.

By condition (ii) ($D_p \downarrow y$), we have $u_{(\mathbf{x},y+1)}' \geq u_{(\mathbf{x},y)}'$ and $u_{(\mathbf{x},y+1)} \geq u_{(\mathbf{x},y)}$. By condition (b), we further have $u_{(\mathbf{x},y)} \geq u_{(\mathbf{x},y)}'$ and $u_{(\mathbf{x},y+1)} \geq u_{(\mathbf{x},y+1)}'$. Hence the vector ($u_{(\mathbf{x},y+1)}'$, $u_{(\mathbf{x},y)}'$, $u_{(\mathbf{x},y+1)}$,

$u_{(\mathbf{x},y)}$) has the following six possible values: $(0,0,0,0)$, $(0,0,1,0)$, $(1,0,1,0)$, $(0,0,1,1)$, $(1,1,0,1)$, and $(1,1,1,1)$. It follows from algebra that (8) analyzed for each of these six cases is ≥ 0.

Next, we show the result for operator T_i^1 and first consider the states away from boundary, i.e., $x_i > 0$.

$$\begin{aligned} &D_p T_i^1 v_k'(\mathbf{x},y) - D_p T_i^1 v_k'(\mathbf{x},y) + R_0 - R_0' \\ &\quad = \max[v_k'(\mathbf{x}-\mathbf{e}_i, y+1) + R_i, v_k'(\mathbf{x}, y+1)] - \max[v_k'(\mathbf{x}-\mathbf{e}_i, y) + R_i, v_k'(\mathbf{x}, y)] \\ &\qquad - \max[v_k(\mathbf{x}-\mathbf{e}_i, y+1) + R_i, v_k(\mathbf{x}, y+1)] + \max[v_k(\mathbf{x}-\mathbf{e}_i, y) + R_i, v_k(\mathbf{x}, y)] + R_0 - R_0' \end{aligned} \tag{9}$$

We redefine w and w' as

$$w(u,\mathbf{x},y) = \begin{cases} v_k(\mathbf{x}-\mathbf{e}_i, y) + R_i, & \text{if } u = 1 \\ v_k(\mathbf{x}, y) & \text{if } u = 0 \end{cases}$$

with w' again defined similar to w, using its corresponding value function v_k'. Using the definitions of u and u' with the new functions w and w', we have $T_i^1 v_k(\mathbf{x},y) = \max_{u\in 0,1} w(u,\mathbf{x},y)$ and $T_i^1 v_k'(\mathbf{x},y) = \max_{u\in 0,1} w'(u,\mathbf{x},y)$.

Condition (i) ($D_i \downarrow y$), implies $u_{(\mathbf{x},y+1)} \geq u_{(\mathbf{x},y)}$ and $u'_{(\mathbf{x},y+1)} \geq u'_{(\mathbf{x},y)}$. Further, by condition (a), we have $u'_{(\mathbf{x},y)} \geq u_{(\mathbf{x},y)}$ and $u'_{(\mathbf{x},y+1)} \geq u_{(\mathbf{x},y+1)}$.

Hence the vector ($u'_{(\mathbf{x},y+1)}$,$u'_{(\mathbf{x},y)}$,$u_{(\mathbf{x},y+1)}$,$u_{(\mathbf{x},y)}$) now has the following six possible values: $(0,0,0,0)$, $(1,0,0,0)$, $(1,1,0,0)$, $(1,0,1,0)$, $(1,1,1,0)$, and $(1,1,1,1)$. It follows from algebra that (9) for each of these cases is ≥ 0.

For the boundary states with $x_i = 0$, only case (0,0,0,0) applies and (9) results in $D_p v_k'(\mathbf{x},y) - D_p v_k(\mathbf{x},y) + R_0 - R_0'$ which is ≥ 0 by condition (b).

For Operator T_0^2, we first consider the states $(\mathbf{x},y)$ for which $x_i > 0\ \forall i$.

$$\begin{aligned} &D_p T_i^1 v_k'(\mathbf{x},y) - D_p T_i^1 v_k'(\mathbf{x},y) + R_0 - R_0' \\ &\quad = \max[v_k'(\mathbf{x}-\mathbf{1}, y+2), v_k'(\mathbf{x}, y+1)] - \max[v_k'(\mathbf{x}-\mathbf{1}, y), v_k'(\mathbf{x}, y)] \\ &\qquad - \max[v_k(\mathbf{x}-\mathbf{1}, y+2), v_k(\mathbf{x}, y+1)] + \max[v_k(\mathbf{x}-\mathbf{1}, y), v_k(\mathbf{x}, y)] + R_0 - R_0' \end{aligned} \tag{10}$$

We redefine w and w' as

$$w(u,\mathbf{x},y) = \begin{cases} v_k(\mathbf{x}-\mathbf{1}, y+1) & \text{if } u = 1 \\ v_k(\mathbf{x}, y) & \text{if } u = 0 \end{cases}$$

with w' defined similar to w. We let $T_0^2 v_k(\mathbf{x},y) = \max_{u\in 0,1} w(u,\mathbf{x},y)$ and $T_0^2 v_k'(\mathbf{x},y) = \max_{u\in 0,1} w'(u,\mathbf{x},y)$. By condition (iii) ($D_{-1,p} \downarrow y$) we have $u'_{(\mathbf{x},y)} \geq u'_{(\mathbf{x},y+1)}$ and $u_{(\mathbf{x},y)} \geq u_{(\mathbf{x},y+1)}$.

Further, by (c), we also have $u_{(\mathbf{x},y)} \geq u'_{(\mathbf{x},y)}$ and $u_{(\mathbf{x},y+1)} \geq u'_{(\mathbf{x},y+1)}$. The vector $(u'_{(\mathbf{x},y+1)}, u'_{(\mathbf{x},y)}, u_{(\mathbf{x},y+1)}, u_{(\mathbf{x},y)})$ therefore has the six possible values: $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,1,1)$, $(0,1,0,1)$, $(0,1,1,1)$, and $(1,1,1,1)$. We analyze (10) for each of these six cases. All cases except for $(0,1,0,1)$ are very similar to the ones analyzed previously, hence we only show that case $(0,1,0,1)$ yields $D_1 v'_k(\mathbf{x} - \mathbf{1}, y+1) - D_1 v_k(\mathbf{x} - \mathbf{1}, y+1) + R_0 - R'_0 \geq 0$ which is implied by jointly by conditions (b) and (c). For the states where $x_i = 0$ for some component i, the only feasible case $(0,0,0,0)$ yields $D_p v'_k(\mathbf{x}, y) - D_p v_k(\mathbf{x}, y) + R_0 - R'_0 \geq 0$ by (b).

The analysis for operator T_i^2 is very similar to the one for operator T_i^1 and is thus omitted. The results hold for operator T as this operator is formed by addition and multiplication of positive constants with the functions $T_n^1 v$ and $T_n^2 v$ for $n = 0,...,N$ and linear inventory holding and assembly queue backorder costs. □

To complete the proof of Theorem 2, we note that conditions (a)-(c) are sufficient for the sensitivity results to hold. For example, condition (a) implies that if $v'_k(\mathbf{x} + \mathbf{e}_i, y) - v'_k(\mathbf{x}, y) > 0$, then $v_k(\mathbf{x} + \mathbf{e}_i, y) - v_k(\mathbf{x}, y) > 0$. Hence, if it is optimal to produce an additional unit of component i at state $(\mathbf{x}, y)$ in problem B, then it is also optimal to produce an additional unit of component i in state $(\mathbf{x}, y)$ in problem A. Therefore $\gamma'_i(\mathbf{x}_{-i}, y) \leq \gamma_i(\mathbf{x}_{-i}, y)$. Through a similar argument, (a) also implies the shift in the component admission threshold $\alpha_i(\mathbf{x}, y)$.

In terms of the product demand admission policy, Condition (ii) implies that if $v'_k(\mathbf{x}, y-1) + R'_0 \geq v_k(\mathbf{x}, y)$, then $v_k(\mathbf{x}, y-1) + R_0 \geq v_k(\mathbf{x}, y)$. Hence, if it is optimal to admit a demand for the end product in problem B, it remains optimal to admit a product demand at that state in problem A resulting in $\beta'(\mathbf{x}) \geq \beta(\mathbf{x})$. As for the optimal assembly policies, analogous arguments yield $\delta'(\mathbf{x}) \leq \delta(\mathbf{x})$. □

Proof of Theorem 3: The proof of Theorem 3 closely follows the steps in the proof of Theorem 1. Therefore, we only provide the sufficient conditions implying the structure of the optimal policy and show that they are preserved across transitions in the following Lemma. We note that, as described in the proof of Theorem 1, without loss of optimality, the problem may be converted to a finite state and finite action space problem. Therefore, a stationary policy for the discounted profit infinite horizon problem exists by Theorem 6.2.10 of Puterman (1994).

Let V be the set of functions defined on the state space such that if $v \in V$, then $\forall\, i, j = 1, \ldots N$ where $j \neq i$;

(i') $D_i v(\mathbf{x}, y) \downarrow x_i, \uparrow x_j, \downarrow y \;\; \forall\, i = 1, \ldots, N$

(ii') $D_p v(\mathbf{x}, y) \downarrow x_i, \downarrow y$, and $\leq R_0$ for $y > 0$

$D_p v(\mathbf{x}, 0) - D_p v(\mathbf{x}, -1) \leq R_0 - r_0$

(iii') $D_{\text{-}\mathbf{1},p} v(\mathbf{x}, y) \uparrow x_i, \downarrow y$

$D_{\text{-}\mathbf{1},p} v(\mathbf{x}, 0) - D_{\text{-}\mathbf{1},p} v(\mathbf{x}, -1) \leq R_0 - r_0$

Conditions (i')-(iii') are similar to the ones (i)-(iii) associated with the original problem except for the conditions given in (ii') and (iii') which enable the optimal policy to hold at the boundary states as well.

Lemma 4. *If $v \in V$ then, $T_n^1 v$, $T_n^2 v$, and $Tv \in V \quad \forall n = 0, \ldots, N$.*

Proof: For brevity, once again we only present the proof for the supermodularity condition given in (i'). We will show that if v satisfies $D_i v(\mathbf{x}, y) \uparrow x_j$, then $T_0^1 v$, $T_i^1 v$, $T_j^1 v$, $T_k^1 v$, $T_0^2 v$, $T_i^2 v$, $T_j^2 v$, $T_k^2 v$, and Tv (where i, j, k are distinct) all satisfy the same condition. As the operators T_0^1 and T_0^2 are the ones that have been modified and the analysis for the remaining operators are similar to the ones in the proof of Lemma 2, we restrict our illustration of the proof for these two operators.

For operator T_0^1 and for $y > 0$, we have $D_i T_0^1 v(\mathbf{x} + \mathbf{e}_j, y) - D_i T_0^1 v(\mathbf{x}, y) = D_i v(\mathbf{x} + \mathbf{e}_j, y - 1) - D_i v(\mathbf{x}, y - 1) \geq 0$ by $D_i v \uparrow x_j$. For $y \leq 0$, we have

$$\begin{aligned} D_i T_0^1 v(\mathbf{x} + \mathbf{e}_j, y) = \max[&v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, y - 1) + r_0, v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, y)] \\ &- \max[v(\mathbf{x} + \mathbf{e}_j, y - 1) + r_0, v(\mathbf{x} + \mathbf{e}_j, y)] \end{aligned} \tag{11}$$

$$D_i T_0^1 v(\mathbf{x}, y) = \max[v(\mathbf{x} + \mathbf{e}_i, y - 1) + r_0, v(\mathbf{x} + \mathbf{e}_i, y)] - \max[v(\mathbf{x}, y - 1) + r_0, v(\mathbf{x}, y)] \tag{12}$$

We need to show that (11) minus (12) ≥ 0. The outcome $v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j, y) - v(\mathbf{x}+\mathbf{e}_j, y-1) - r_0$ is not feasible for (11) due to $D_p \downarrow x_i$ in (ii'). The three remaining feasible outcomes are $D_i v(\mathbf{x}+\mathbf{e}_j, y-1)$, $v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, y - 1) - v(\mathbf{x} + \mathbf{e}_j, y)$, and $D_i v(\mathbf{x} + \mathbf{e}_j, y)$ for which simple algebra yields (11) minus (12) ≥ 0 in all three cases.

For operator T_0^2 we only show the result for $y < 0$ as the cases for $y \geq 0$ are identical to the ones analyzed in Lemma 2. For $x_i > 0\; \forall i$, we have

$$\begin{aligned} D_i T_0^2 v(\mathbf{x} + \mathbf{e}_j, y) = \max[&v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{1}, y + 1) + R_0 - r_0, v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, y)] \\ &- \max[v(\mathbf{x} + \mathbf{e}_j - \mathbf{1}, y + 1) + R_0 - r_0, v(\mathbf{x} + \mathbf{e}_j, y)] \end{aligned} \tag{13}$$

$$\begin{aligned} D_i T_1^1 v(\mathbf{x}, y) = \max[&v(\mathbf{x} + \mathbf{e}_i - \mathbf{1}, y + 1) + R_0 - r_0, v(\mathbf{x} + \mathbf{e}_i, y)] \\ &- \max[v(\mathbf{x} - \mathbf{1}, y + 1) + R_0 - r_0, v(\mathbf{x}, y)] \end{aligned} \tag{14}$$

Eliminating the infeasible outcomes due to (iii'), each of the remaining cases where (13) results in either $D_i v(\mathbf{x}+\mathbf{e}_j-\mathbf{1}, y+1)$, $v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j-\mathbf{1}, y+1)+R_0-r_0-v(\mathbf{x}+\mathbf{e}_j, y)$, or $D_i v(\mathbf{x}+\mathbf{e}_j, y+1)$ results in (13) minus (14) ≥ 0 through simple algebra. Similar steps also yields the results for boundary states where $x_i = 0$, $x_j = 0$, or $x_k = 0$. $\square$

References

Akcay, Y., Xu, S. H. (2004) Joint inventory replenishment and component allocation optimization in an assemble-to-order system. *Management Science*, **50**, 99-116.

Benjaafar, S., ElHafsi, M. (2006) Production and inventory control of a single product assemble-to-order system with multiple customer classes. *Management Science*, **52**, 1896-1912.

Benjaafar, S., ElHafsi M., Lee, C. Y., and Zhou, W. (2010) Optimal control of assembly systems with multiple stages and multiple demand classes. Forthcoming, *Operations Research*.

Billinton, R., Allan, R.N. (1983) *Reliability Evaluation of Engineering Systems: Concepts & Techniques*. Plenum Press, New York, NY.

Carr, S., Duenyas, I. (2000) Optimal admission control and sequencing in a make-to-stock/make-to-order production system. *Operations Research*, **48**, 709-720.

Cohen, M. A., Agrawal, N., Agrawal, V. (2006) Winning in the aftermarket. *Harvard Business Review*, May 2006.

Duenyas, I., Tsai, C. (2001) Centralized and decentralized control of a two stage tandem manufacturing system with demand for intermediate and end product. *Working paper*. University of Michigan, Ann Arbor, MI.

Gerchak, Y., Henig, M. (1989) Component commonality in assemble-to-order systems: Models and properties. *Naval Reseach Logistics*, **36**, 61-68.

Ghoneim, H., Stidham, S. (1985) Control of arrivals to two queues in series. *European Journal of Operational Research*, **21**, 399-409.

Ha, A. (1997a) Optimal dynamic scheduling policy for a make-to-stock production system. *Operations Research*, **45**, 42-53.

Ha, A. (1997b) Inventory rationing in a make-to-stock production system with several demand classes and lost sales. *Management Science*, **43**, 1093-1103.

Hausman, W. H., Lee, H. L., Zhang, A. X. (1998) Joint demand fulfillment probability in a multi-item inventory system with independent order-up-to policies. *European Journal of Operational Research*, **109**, 646-659.

Koole, G. (2006) Monotonicity in Markov reward and decision chains: Theory and applications. *Foundation and Trends in Stochastic Systems*, **1**(1), 176.

Ku, C.Y., Jordan, S. (1997) Access control to two multiserver loss queues in series. *IEEE Transactions on Automatic Control*, **42**, 1017-1023.

Ku, C.Y., Jordan, S. (2002) Access control of parallel multiserver loss queues. *Performance Evaluation*, **50**, 219-231.

Lippman, S.A. (1975) Applying a new device in the optimization of exponential queuing systems. *Operations Research*, **23**, 687-710.

Puterman, M. L. (1994) *Markov Decision Processes*, John Wiley and Sons, Inc., New York, NY.

Rosling, K. (1989) Optimal inventory policies for assembly systems under random demands. *Operations Research*, **37**, 565-579.

Schmidt, C., Nahmias, S. (1985) Optimal policy for a two-stage assembly system under random demand. *Operations Research*, **33**, 1130-1145.

Song, J. S., Xu, S. H., Liu, B. (1999) Order-fulfillment performance measures in an assemble-to-order system with stochastic leadtimes. *Operations Research*, **47**, 131-149.

Song, J. S., Zipkin, P. (2003) Supply chain operations: Assembly-to-order systems in *Handbooks in OR & MS*, **47**, Elsevier B.V.

Stidham, S. (1985) Optimal control of admission to a queueing system. *IEEE Trans. Automatic Control*, **30**(8), 705-713.

Tijms, H. C. (1986) *Stochastic Modeling and Analysis: A Computational Approach.* John Wiley and Sons, Inc., New York, NY.

Topkis, D. M. (1998) *Supermodularity and Complementarity.* Princeton University Press, New Jersey, NJ.

Veatch, M.H., Wein, L.M. (1994) Optimal control of a two-station tandem production/inventory system. *Operations Research*, **42**, 337-350.